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COMPARISON OF BASIC DESIGNS OF INTERVAL OBSERVERS FOR LINEAR TIME-INVARIANT SYSTEMS

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Аннотация: Several basic and popular schemes of design of interval observers for linear autonomous dynamical systems are summarized and compared through their conditions of applicability and provided quality of estimation.

1. Introduction

The problem of state vector estimation for a dynamical system can be encountered in many engineering applications and other domains of science [1–3].

In the presence of uncertainty in the model of a dynamical process, design of a conventional state estimator, which has to converge to the ideal value of the state, may be rarely realized. In such a case, in general, the state estimation error is not approaching zero (it can be bounded or asymptotically bounded, and different versions of practical stability are used for analysis). A straightforward example deals with the presence of the measurement noise, in such a situation the state estimate usually contains a noise shift providing the static errors. In this case, an interval or set-membership estimation is often more feasible: an observer can be constructed that, using input-output information and the bounds on the uncertain elements, evaluates a bounded set of admissible values (interval) for the state at each instant of time. It is desirable that the interval length be minimized by tuning the observer parameters, and it should be proportional to the size of the model uncertainty.

There are many methods to design interval estimators, see surveys [4], and in this note we are going to recall and compare several popular solutions obtained in the literature for the simplest case of linear time-invariant system.

Notation

The symbols I_n and E_p denote the identity matrix with dimension $n \times n$ and the vector with all elements equal 1 of dimension p. For two vectors $x_1, x_2 \in \mathbb{R}^n$, the relations $x_1 \leqslant x_2$ are understood elementwise. Given a matrix $A \in \mathbb{R}^{m\times n}$, define $A^+ = \max\{0, A\}$,

$A^- = A^+ - A$ (similarly for vectors) and denote the matrix of absolute values of all elements by $|A| = A^+ + A^-$.

2. Benchmark problem

Consider the following system

$$\dot{x}(t) = Ax(t) + d(t),\ y(t) = Cx(t) + v(t),\ t \in \mathbb{R}_+ \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the output; $d(t) \in \mathbb{R}^n$ is a bounded disturbance; $v(t) \in \mathbb{R}^p$ is a bounded measurement noise; the matrices A, C have appropriate dimensions. This model has three sources of uncertainty (initial conditions for $x(0)$, instant values of d and v), and the standard standing hypotheses are that all of them belong to known intervals:

Предположение 1. *Let $x(0) \in [\underline{x}_0, \overline{x}_0]$ for some known $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$, let also two functions $\underline{d}, \overline{d} \in \mathcal{L}_\infty^n$ and a constant $V > 0$ be given such that*

$$\underline{d}(t) \leqslant d(t) \leqslant \overline{d}(t),\ |v(t)| \leqslant V \quad \forall t \geqslant 0.$$

It is required to calculate two estimates $\underline{x}(t), \overline{x}(t) \in \mathbb{R}^n$ with a bounded discrepancy $\overline{x} - \underline{x}$, using the available information about uncertainty and $y(t)$, such that

$$\underline{x}(t) \leqslant x(t) \leqslant \overline{x}(t) \quad \forall t \geqslant 0. \tag{2}$$

3. Description of design methods

In this section we assume that Assumption 1 holds, and summarize several design methods for interval observers solving the above problem.

3.1. Basic structure

The following is the simplest interval observer:

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + L[y(t) - C\underline{x}(t)] - |L|E_pV + \underline{d}(t), \\ \dot{\overline{x}}(t) &= A\overline{x}(t) + L[y(t) - C\overline{x}(t)] + |L|E_pV + \overline{d}(t), \\ \underline{x}(0) &= \underline{x}_0,\ \overline{x}(0) = \overline{x}_0, \end{aligned} \tag{3}$$

where $L \in \mathbb{R}^{n\times p}$ is the observer gain to be designed. The conditions to satisfy for L are given below.

Теорема 1. [5] *In the system (1) with the interval observer (3) the relations (2) are satisfied provided that the matrix $A - LC$ is Metzler. In addition, $\overline{x} - \underline{x}$ is bounded if $A - LC$ is Hurwitz.*

Thus, the matrix L should be chosen in a way providing Metzler and Hurwitz properties for $A - LC$. In in order to minimize the width of estimated interval $[\underline{x}(t), \overline{x}(t)]$, the L_1 optimization problem can be formulated as a linear program [6]: it is necessary to find $\lambda \in \mathbb{R}^n$, $w \in \mathbb{R}^p$ and a diagonal matrix $M \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned} &\begin{bmatrix} A^\top\lambda - C^\top w + E_n \\ \lambda - \gamma E_n \end{bmatrix} < 0, \\ &\lambda > 0,\ M \geqslant 0, \\ &A^\top\lambda - C^\top w + M\lambda \geqslant 0, \end{aligned} \tag{4}$$

then $w = L^\top\lambda$.

3.2. Transformations of coordinates to nonnegative form

Program (4) in some cases may have no solution, then to overcome the issue a transformation of coordinates can be used. We can design the gain L such that the matrix $A - LC$ is Hurwitz, and next find a similarity transformation $S \in \mathbb{R}^{n\times n}$ such that $D = S(A - LC)S^{-1}$ is Metzler (it is Hurwitz by construction). The conditions of existence of such a matrix S are given in [7]. Next, in the new coordinates $z = Sx$ the system (1) takes the form:

$$\dot{z}(t) = Dz(t) + SLy(t) + \delta(t),\ \delta(t) = S[d(t) - Lv(t)], \tag{5}$$

where $\underline{\delta}(t) \leqslant \delta(t) \leqslant \overline{\delta}(t)$ with $\underline{\delta}(t) = S^+\underline{d}(t) - S^-\overline{d}(t) - |SL|E_pV$ and $\overline{\delta}(t) = S^+\overline{d}(t) - S^-\underline{d}(t) + |SL|E_pV$. For the system (5) all conditions of Theorem 1 are satisfied and an interval observer similar to (3) can be designed:

$$\begin{aligned} &\dot{\underline{z}}(t) = D\underline{z}(t) + SLy(t) + \underline{\delta}(t), \\ &\dot{\overline{z}}(t) = D\overline{z}(t) + SLy(t) + \overline{\delta}(t), \\ &\underline{z}(0) = S^+\underline{x}_0 - S^-\overline{x}_0,\ \overline{z}(0) = S^+\overline{x}_0 - S^-\underline{x}_0, \\ &\underline{x}(t) = (S^{-1})^+\underline{z}(t) - (S^{-1})^-\overline{z}(t), \\ &\overline{x}(t) = (S^{-1})^+\overline{z}(t) - (S^{-1})^-\underline{z}(t). \end{aligned} \tag{6}$$

In (6) the inclusion (2) is satisfied and $\overline{x} - \underline{x} = |S^{-1}|(\overline{z} - \underline{z})$ is bounded.

3.3. Time-varying transformation of coordinates

If there is no matrix S, then it is possible to consider a time-varying transformation of coordinates:

Лемма 1. [8] *Let $A-LC$ be Hurwitz, then there exists an invertible matrix function $P : \mathbb{R} \to \mathbb{R}^{n\times n}$, of class C^∞ elementwise, $||P(t)||_2 < +\infty$ for all $t \in \mathbb{R}$, such that for all $t \in \mathbb{R}$*

$$\dot{P}(t) = DP(t) - P(t)(A - LC),$$

where $D \in \mathbb{R}^{n\times n}$ is a Hurwitz and Metzler matrix.

Under conditions of Lemma 1, in the new coordinates $\zeta(t) = P(t)x(t)$ we obtain

$$\dot{\zeta}(t) = D\zeta(t) + P(t)Ly(t) + \delta(t),\ \delta(t) = P(t)[d(t) - Lv(t)]$$

and again

$$\underline{\delta}(t) \leqslant \delta(t) \leqslant \overline{\delta}(t),$$
$$\underline{\delta}(t) = P^+(t)\underline{d}(t) - P^-(t)\overline{d}(t) - |P(t)L|E_pV,$$
$$\overline{\delta}(t) = P^+(t)\overline{d}(t) - P^-(t)\underline{d}(t) + |P(t)L|E_pV.$$

The interval observer has a form similar to (6) [8]:

$$\begin{aligned}
&\dot{\underline{z}}(t) = D\underline{z}(t) + P(t)Ly(t) + \underline{\delta}(t),\\
&\dot{\overline{z}}(t) = D\overline{z}(t) + P(t)Ly(t) + \overline{\delta}(t),\\
&\underline{z}(0) = P^+(t)\underline{x}_0 - P^-(t)\overline{x}_0,\ \overline{z}(0) = P^+(t)\overline{x}_0 - P^-(t)\underline{x}_0,\\
&\underline{x}(t) = (P^{-1})^+(t)\underline{z}(t) - (P^{-1})^-(t)\overline{z}(t),\\
&\overline{x}(t) = (P^{-1})^+(t)\overline{z}(t) - (P^{-1})^-(t)\underline{z}(t),
\end{aligned} \tag{7}$$

however, its realization needs more computations than for (6) since the obtained interval estimator is time-varying. It is also difficult to optimize the width of the estimated interval.

3.4. Intermediate solution

There is another approach presented in [9] that mixes the advantages of a (partial) static transformation of coordinates with simplicity of formulation in the original variables, where the following interval observer is proposed:

$$\begin{aligned}
&\underline{x}(t) = \underline{z}(t) + Ny(t) - |N|E_pV,\\
&\overline{x}(t) = \overline{z}(t) + Ny(t) + |N|E_pV,\\
&\dot{\underline{z}}(t) = (TA - LC)\underline{x}(t) + Ly(t)\\
&-(2(TA - LC)^-|N| + |L|)E_pV + T^+\underline{d}(t) - T^-\overline{d}(t),\\
&\dot{\overline{z}}(t) = (TA - LC)\overline{x}(t) + Ly(t)\\
&+(2(TA - LC)^-|N| + |L|)E_pV + T^+\overline{d}(t) - T^-\underline{d}(t),\\
&\underline{z}(0) = T^+\underline{x}_0 - T^-\overline{x}_0,\ \overline{z}(0) = T^+\overline{x}_0 - T^-\underline{x}_0,
\end{aligned} \tag{8}$$

where $L, N \in \mathbb{R}^{n\times p}$ are observer gains to select and $T = I_n - NC$.

Теорема 2. *In the system (1) with the interval observer (8) the relations (2) are satisfied provided that the matrix $TA - LC$ is Metzler. In addition, $\overline{x} - \underline{x}$ is bounded if $TA - LC$ is Hurwitz.*

For $N = 0$ (*i.e.*, $T = I_n$) we recover the case of Theorem 1. Thus, (8) is a natural extension of the observer (3) since the conditions of Theorem 2 impose the restrictions on the matrix $TA - LC = A - LC - NCA$, where by the choice of N it is possible to recover the Metzler property even in the case there is no matrix L providing this feature to the matrix $A - LC$.

The linear program (4) takes the following form in this case: it is necessary to find $\lambda \in \mathbb{R}^n$, $w_1, w_2 \in \mathbb{R}^p$ and a diagonal matrix $M \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned}
&\begin{bmatrix} A^\top\lambda - C^\top w_1 - A^\top C^\top w_2 + E_n \\ \lambda - \gamma E_n \end{bmatrix} < 0,\\
&\lambda > 0,\ M \geqslant 0,\\
&A^\top\lambda - C^\top w_1 - A^\top C^\top w_2 + M\lambda \geqslant 0,
\end{aligned} \tag{9}$$

then $w_1 = L^\top\lambda$ and $w_2 = N^\top\lambda$.

3.5. Nonnegative embedding

It is worth to mention that in general, any system may be immersed in its internal positive representation [10], which has dimension $2n$. For example, any matrix can be decomposed as a difference of Metzler and nonnegative matrices:

$$\begin{aligned} A - LC &= (A-LC)_{\diagdown} + (A-LC)_{\times} \\ &= \underbrace{(A-LC)_{\diagdown} + (A-LC)_{\times}^{+}}_{\text{Metzler}} - \underbrace{(A-LC)_{\times}^{-}}_{\text{nonnegative}}, \end{aligned}$$

where $(A-LC)_{\diagdown}$ is the diagonal matrix composed by the elements of $A-LC$ on the main diagonal and $(A-LC)_{\times} = A - LC - (A-LC)_{\diagdown}$. Then the following interval observer can be proposed for (1):

$$\begin{aligned} \dot{\underline{x}}(t) &= [(A-LC)_{\diagdown} + (A-LC)_{\times}^{+}]\underline{x}(t) - (A-LC)_{\times}^{-}\overline{x}(t) \\ &\quad + Ly(t) - |L|E_pV + \underline{d}(t), \\ \dot{\overline{x}}(t) &= [(A-LC)_{\diagdown} + (A-LC)_{\times}^{+}]\overline{x}(t) - (A-LC)_{\times}^{-}\underline{x}(t) \\ &\quad + Ly(t) + |L|E_pV + \overline{d}(t), \\ &\quad \underline{x}(0) = \underline{x}_0,\ \overline{x}(0) = \overline{x}_0. \end{aligned} \tag{10}$$

Теорема 3. [11] *In the system (1) with the interval observer (10) the relations (2) are satisfied. In addition,* $\overline{x} - \underline{x}$ *is bounded if*

$$R = \begin{bmatrix} A-LC)_{\diagdown} + (A-LC)_{\times}^{+} & (A-LC)_{\times}^{-} \\ (A-LC)_{\times}^{-} & A-LC)_{\diagdown} + (A-LC)_{\times}^{+} \end{bmatrix}$$

is Hurwitz.

As we can conclude the requirement on Metzler property of the matrix $A - LC$ is completely avoided, and the main difficulty in application of the last theorem consists in finding conditions under which the Metzler matrix R is Hurwitz. It is also difficult to formulate some LMIs to find L in this setting.

4. Comparison

For comparison purposes, the observers (3) and (6) (the latter is a variant of the former after a state transformation) can be considered together, since they represent the first milestone idea of interval observer design. Other solutions are an observer based on time-varying transformation of coordinates (7), NL-design (8), and nonnegative representation-based scheme (10). To simplify a comparative evaluations of these interval observers, the following table is given:

Observer	Advantages	Disadvantages	CC
(6)	CTC (4)	May do not exist	Low
(7)	Always exist	No CTC	High
(8)	CTC (9)	May do not exist	Low
(10)	Unstable version always exist	No CTC	Low

where CTC = Computationally Tractable Conditions, CC = Computational Complexity

5. Conclusion

If it is possible to find L (and N) such that $A - LC$ ($TA - LC$) is Hurwitz and Metzler (looking for a solution of (4) or (9)), then the interval observer (3) or (8) is the right choice: they have low computational complexity of implementation and allow estimation accuracy to be optimized. If such a gain L does not exist, then different transformations of coordinates can be tested (interval observer (6) is a natural extension). In the time-varying case, it is enough to get $A-LC$ to be Hurwitz (as in the conventional point-wise estimation), but optimization of the gain becomes difficult in this case, while implementation requires a lot of calculations. An alternative solution (10) also can be tested, which always guarantees the interval inclusion (2), but the boundedness of the estimated interval $\overline{x} - \underline{x}$ can be an issue.

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